Exercise 1:
Let \( I \subseteq k[x_1, \ldots, x_n] \) be a principal ideal (that is, \( I \) is generated by a single \( f \in I \)). Show that any finite subset of \( I \) containing a generator for \( I \) is a Groebner basis for \( I \).

Exercise 2:
Let \( f, g \in k[x_1, \ldots, x_n] \) be polynomials such that \( \text{LM}(f) \) and \( \text{LM}(g) \) are relatively prime monomials and \( \text{LC}(f) = \text{LC}(g) = 1 \). Show that
\[
S(f, g) = -(g - \text{LT}(g))f + (f - \text{LT}(f))g.
\]

Exercise 3:
Consider an ideal \( I \) generated by \( I = \langle xz - y, xy + 2z^2, y - z \rangle \). Is this generating set a Groebner basis for \( I \)? If not, find a Groebner basis. What will be a minimal and the reduced Groebner basis for \( I \)?

Exercise 4:
Let \( G \) be a Groebner basis of an ideal \( I \) with the property that \( \text{LC}(g) = 1 \) for all \( g \in G \). Prove that \( G \) is a minimal Groebner basis if and only if no proper subset of \( G \) is a Groebner basis.

Exercise 5:
Let \( G \) and \( G' \) be Groebner bases for an ideal \( I \) with respect to the same monomial order in \( k[x_1, \ldots, x_n] \). Show that \( \bar{f}^G = \bar{f}^{G'} \) (here we write \( \bar{f}^G \) for the remainder on division of \( f \) by a Groebner basis \( G = \langle f_1, \ldots, f_s \rangle \)). Hence, the remainder of division by a Groebner basis is independent of which Groebner basis we use, as long as we fix a monomial order.